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SANDWICH-TYPE THEOREMS FOR MEROMORPHIC MULTIVALENT FUNCTIONS ASSOCIATED WITH A LINEAR OPERATOR

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Abstract— the purpose of this article is to obtain some subordination and superordination preserving properties of meromorphic multivalent functions in the punctured open unit disk associated with the linear operator $Q^{p}_{\alpha,\beta,\gamma}$ the sandwish- type results for these meromorphic multivalent functions are also considered.

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1 Introduction

LET $H(\mathrm{U})$ denote the class of analytic functions in the open unit disk $\mathrm{U}=\{z\in\mathrm{C}: |z|<1\}$.

For $n\in\mathrm{N}=\{1,2,\ldots\}$ and $a\in\mathrm{C}$, let $H[a,n]=\{f\in H \;:\; f(z)=a+a_n z^n+a_{n+1}z^{n+1}+\ldots\}$.

Let f and F be members of H. the function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w analytic in U, with $w(0)=0$ and $|w(z)|<1 \quad (z\in\mathrm{U})$, such that

$$f(z)=F(w(z)) \quad (z\in\mathrm{U}).$$

In such a case, we write

$$f\prec F \quad (z\in\mathrm{U}) \quad \text{or} \quad f(z)\prec F(z) \quad (z\in\mathrm{U}).$$

If the function F is univalent in U, then we have (cf. [11])

$$f(z)\prec F(z) \quad (z\in\mathrm{U}) \Leftrightarrow f(0)=F(0)$$

$$\text{and} \quad f(\mathrm{U})\subset F(\mathrm{U}).$$

Definition 1 [9]. Let $\phi:\mathrm{C}^2\to\mathrm{C}$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination:

$$\phi(p(z),zp'(z))\prec h(z) \quad (z\in\mathrm{U}), \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p\prec q$ for all p satisfying (1.1). A domnant that satisfies for all dominants q of (1.1) is said to be the best dominant.

Definition 2 [10]. Let $\varphi:\mathrm{C}^2\to\mathrm{C}$ and let h be analytic in U. If p and $\varphi(p(z),zp'(z))$ are univalent in U and satisfy the differential superordination:

$$h(z)\prec\varphi(p(z),zp'(z)) \quad (z\in\mathrm{U}), \tag{1.2}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the soltions of the differential superordination, or more simply a subordinant if $q\prec p$ for all p satisfying (1.2). A univlent subordinant that for all subordinants q of (1.2) is said to be the best subordinant.

Definition 3 [10]. Denote by Q the class of functions f that are analytic and injective on $\overline{\mathrm{U}}\setminus E(f)$, where

$$E(f)=\left\{\zeta\in\partial U \;:\; \lim_{z\to\zeta} f(z)=\infty\right\},$$

and are such that

$$f'(\zeta)\neq 0\left(\zeta\in\partial\mathrm{U}\setminus E(f)\right).$$

For any integer $m>-p$, Let $\Sigma_{p,m}$ denote the class of all meromorphic functions f of the form

$$f(z)=z^{-p}+\sum_{k=m}^{\infty}a_k z^k \qquad (p\in\mathrm{N}=\{1,2,\ldots\}), \tag{1.3}$$

which are analytic and p-valent in the punctured unit disk $\mathrm{U}^*=\{z\in\mathrm{C}:\; 0<|z|<1\}=\mathrm{U}\setminus\{0\}$. For convenience, we write $\Sigma_{p,1-p}=\Sigma_p$.

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For functions $f \in \Sigma_{p,m}$ given by (1.3), and $g \in \Sigma_{p,m}$ defined by

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k \quad (m > -p; p \in \mathrm{N}), \qquad (1.4)$$

then the Hadamard product (or convolution) of f and g is

$$(f * g)(z) = z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k = (g * f)(z) \qquad (1.5)$$

$$(m > -p; p \in \mathrm{N}).$$

For $f \in \Sigma_{p,m}$ we now define the integral operator $Q^p_{\alpha,\beta,\gamma} : \Sigma_{p,m} \to \Sigma_{p,m}$ which was introduced and studied by El-Ashwah et al. [3] as follows:

$$Q^p_{\alpha,\beta,\gamma} f(z) = \frac{\Gamma(\alpha+\beta-\gamma+1)}{\Gamma(\beta)\Gamma(\alpha-\gamma+1)} \frac{1}{z^{\beta+p}} \cdot \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-\gamma} t^{\beta+p-1} f(t)dt$$

$$= \frac{1}{z^p} + \frac{\Gamma(\alpha+\beta-\gamma+1)}{\Gamma(\beta)} \cdot \sum_{k=m}^{\infty} \frac{\Gamma(\beta+p+k)}{\Gamma(\alpha+\beta+p+k-\gamma+1)} a_k z^k \qquad (1.6)$$

$$(\beta > 0;\ \alpha > \gamma - 1;\ \gamma > 0;\ p \in \mathrm{N};\ z \in \mathrm{U}^*),$$

and

$$Q^p_{\alpha-1,\beta,\gamma} f(z) = f(z) \ \ (\beta > 0;\ \gamma > 0;\ p \in \mathrm{N};\ z \in \mathrm{U}^*).$$

From (1.6), it is easy to verify that

$$z(Q^p_{\alpha+1,\beta,\gamma} f(z))' = (\alpha+\beta-\gamma+1) Q^p_{\alpha,\beta,\gamma} f(z) - (\alpha+\beta+p-\gamma+1) Q^p_{\alpha+1,\beta,\gamma} f(z). \qquad (1.7)$$

Remark:

(i) For $\gamma = 1$, $Q^p_{\alpha,\beta,1} = Q^p_{\alpha,\beta}$, where the operator $Q^p_{\alpha,\beta}$ was introduced and studied by Aqlan et al. [2] (see also [1]);

(ii) For $p = \gamma = 1$, $Q^1_{\alpha,\beta,1} = Q^p_{\alpha,\beta}$, where the operator $Q_{\alpha,\beta}$, was introduced and studied by Lashin [6].

2. A Set of Lemms

The following lemmas will be required in our present investigation.

Lemma 1 [9]. Let $p \in Q$ with $p(0) = a$ and let

$$q(z) = a + a_n z^n + ...$$

be analytic in U with

$$q(z) \neq a \ \text{ and } \ n \geq 1.$$

If is not subordinate to , then there exist points

$$z_0 = r_0 e^{i\theta} \in \mathrm{U} \ \text{ and } \ \zeta_0 \in \partial\mathrm{U} \setminus E(f),$$

for which

$$q(\mathrm{U}_{r_0}) \subset p(\mathrm{U}),\ q(z_0) = p(\zeta_0)$$

$$\text{and } \ z_0 q'(z_0) = m\zeta_0 p'(\zeta_0) \quad (m \geq n).$$

A function $L(z,t)$ defined on $\mathrm{U} \times [0,\infty)$, is the subordination chain (or Löwner chain) if $L(z,.)$ is analytic and univalent in U for all $t \in [0,\infty)$, $L(z,.)$ is continuously Differentiable on $[0,\infty)$ for all $z \in \mathrm{U}$ and $L(z,s) \prec L(z,t)$ $(z \in \mathrm{U}; 0 \leq s < t)$.

Lemma 2 [10]. Let $q \in \mathbf{H}[a,1]$ and $\varphi : \mathrm{C}^2 \to \mathrm{C}$. Also set

$$\varphi(q(z), zq'(z)) \equiv h(z) \qquad (z \in \mathrm{U}).$$

If $L(z,t) = \varphi(q(z),\ tzq'(z))$ is a subordination chain and $p \in \mathbf{H}[a,1] \cap Q$, then.

$$h(z) \prec \varphi(p(z), zp'(z)) \qquad (z \in \mathrm{U}).$$

implies that

Furthermore, if $\varphi(q(z), zp'(z)) = h(z)$ has a univalent solution $q \in Q$, then q is the best subordinate.

Lemma 3 [7]. Suppose that the function $H : \mathrm{C}^2 \to \mathrm{C}$ satisfies the following condition:

$$\mathrm{Re}\{H(is,t)\} \leq 0$$

for all real s and

$$t \leq -n(1+s^2)/2 \quad (n \in \mathrm{N}).$$

If the function $p(z) = 1 + p_n z^n + ...$ is analytic in U and

$$\mathrm{Re}\{H(p(z), zp'(z))\} > 0 \ \ (z \in \mathrm{U}),$$

then

$$\mathrm{Re}\{p(z)\} > 0 \ \ (z \in \mathrm{U}).$$

Lemma 4 [8]. Let $\beta, \gamma \in \mathrm{C}$ with $\beta \neq 0$ and $h \in \mathbf{H}(\mathrm{U})$ with $h(0) = c$. If

$$\mathrm{Re}\{\beta h(z) + \gamma\} > 0 \ \ (z \in \mathrm{U}),$$

then, the solution of the differential equation

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \quad (z \in \mathrm{U};\ q(0) = c),$$

is analytic in U satisfies the inequality

$$\mathrm{Re}\{\beta q(z) + \gamma\} > 0 \ \ (z \in \mathrm{U}).$$

Lemma 5 [11]. The function $L(z,t) = a_1(t)z + ...$ with

$a_1(t) \neq 0$ and $\lim_{t\to\infty}|a_1(t)| = \infty$ is a subordination chain if and only if

$$\mathrm{Re}\left\{\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} > 0 \quad (z \in \mathrm{U};\ 0 \le t < \infty),$$

and

$$|L(z;t)| \le K_0|a_1(t)|,\ |z| < r_0 < 1, t \ge 0,$$

For some positive constants K_0 and r_0, then $L(z,t)$ is a subordination chain.

3. Main Results

We begin with proving the following subordination theorem involving the operator $Q^p_{\alpha,\beta,\gamma} f(z)$ defined by (1.6).

Theorem 1. Let $f, g \in \Sigma_{p,m}$, $\alpha > \gamma$, $\beta > 0$, $\gamma > 0$, $p \in \mathrm{N}$, $0 \le \eta < p$ and $z \in \mathrm{U}$. Suppose that

$$\mathrm{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta, \tag{3.1}$$

setting

where

$$\delta = \frac{(p-\eta)^2 + p^2(\alpha+\beta-\gamma)^2}{4p(p-\eta)(\alpha+\beta-\gamma)} - \frac{\left|(p-\eta)^2 - p^2(\alpha+\beta-\gamma)^2\right|}{4p(p-\eta)(\alpha+\beta-\gamma)}\ (0 < \delta \le 1/2). \tag{3.2}$$

Then, the following subordination relation

(3.3)

implies that

$$z^p Q^p_{\alpha,\beta,\gamma} f(z) \prec z^p Q^p_{\alpha,\beta,\gamma} g(z) \quad (z \in \mathrm{U}). \tag{3.4}$$

Moreover, the function $z^p Q^p_{\alpha,\beta,\gamma} g(z)$ is the best dominant.

Proof. Let us define the functions F and G, respectively, by

$$F(z) := z^p Q^p_{\alpha,\beta,\gamma} f(z) \ \text{ and } \ G(z) := z^p Q^p_{\alpha,\beta,\gamma} g(z). \tag{3.5}$$

We first show. If the function is defined by

$$q(z) := 1 + \frac{zG''(z)}{G'(z)} \quad (z \in \mathrm{U}), \tag{3.6}$$

then,

$$\mathrm{Re}\{q(z)\} > 0 \quad (z \in \mathrm{U}).$$

Taking the logarithmic differentiation on both sides of the second equation in (3.5) and using the equation (1.7) we obtain

$$p(\alpha+\beta-\gamma)\phi(z) = p(\alpha+\beta-\gamma)G(z) + (p-\eta)zG'(z). \tag{3.7}$$

Now, by differentiating both sides of (3.7), We obtain the relationship:

$$1 + \frac{z\phi''(z)}{\phi'(z)} = 1 + \frac{zG''(z)}{G'(z)} + \frac{zq'(z)}{q(z) + p(\alpha+\beta-\gamma)/(p-\eta)}$$
$$= q(z) + \frac{zq'(z)}{q(z) + p(\alpha+\beta-\gamma)/(p-\eta)} \equiv h(z). \tag{3.8}$$

We also note from (3.1) that

$$\mathrm{Re}\left\{h(z) + \frac{p(\alpha+\beta-\gamma)}{p-\eta}\right\} > 0 \quad (z \in \mathrm{U}),$$

and so by Lemma 4, we conclude that the differential equation (3.8) has a solution $q \in \mathrm{H}(\mathrm{U})$ with

$$q(0) = h(0) = 1.$$

Let us put

$$H(u,v) = u + \frac{v}{u + p(\alpha+\beta-\gamma)/(p-\eta)} + \delta \tag{3.9}$$

where δ is given by (3.2).
From (3.1), (3.8) and (3.9), we obtain

Now, we proceed to show that

$$\mathrm{Re}\{H(is,t)\} \le 0 \quad \left(s \in \mathrm{R};\ t \le -\frac{1}{2}(1+s^2)\right). \tag{3.10}$$

Indeed, From (3.9), we have

$$\mathrm{Re}\{H(is,t)\} = \mathrm{Re}\left\{is + \frac{t}{is + p(\alpha+\beta-\gamma)/(p-\eta)} + \delta\right\}$$
$$= \frac{tp(\alpha+\beta-\gamma)/p-\eta}{|p(\alpha+\beta-\gamma)/(p-\eta) + is|^2} + \delta$$
$$\le -\frac{E_\delta(s)}{2|p(\alpha+\beta-\gamma)/(p-\eta) + is|^2}, \tag{3.11}$$

where

$$E_\delta(s) := \left(\frac{p(\alpha+\beta-\gamma)}{p-\eta} - 2\delta\right)s^2 - \frac{p(\alpha+\beta-\gamma)}{p-\eta}\cdot \left(2\delta\frac{p(\alpha+\beta-\gamma)}{p-\eta} - 1\right). \tag{3.12}$$

For δ given by (3.2), we can prove easily that the expression $E_\delta(s)$ given by (3.12) is greater than or equal to zero. Hence, from (3.9), we see that (3.10) holds true. Thus using Lemma 3, we conclude that.

$$\mathrm{Re}\{q(z)\} > 0 \quad (z \in \mathrm{U}).$$

Moreover, we see that the condition:

$$G'(0) \neq 0$$

is satisfied. Hence, the function defined by (3.5) is convex in U.

Next, we prove that the subordination condition (3.3) implies that

$$F(z) \prec G(z) \qquad (z \in \mathrm{U}) \tag{3.13}$$

for the functions F and G defined by (3.5). Without loss of generality, we can assume that G is analytic and univalent on $\overline{\mathrm{U}}$ and

$$G'(\zeta) \neq 0 \ \ (\zeta \in \partial \mathrm{U}).$$

For this purpose, we consider the function $L(z,t)$ given by

$$L(z,t) := G(z) + \frac{(p-\eta)(1+t)}{p(\alpha+\beta-\gamma)} zG'(z)$$

$$(0 \le t < \infty;\ 0 \le \eta < p;\ z \in \mathrm{U}).$$

We note that

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0) \cdot$$

$$\left(\frac{p(\alpha+\beta-\gamma)+(p-\eta)(1+t)}{(\alpha+\beta-\gamma)}\right) \neq 0.$$

This shows that the function

$$L(z,t) = \alpha_1(t) z + \ldots$$

satisfies the condition

$$\alpha_1(t) \neq 0 \quad (0 \le t < \infty).$$

Furthermore, we have

$$\mathrm{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} =$$

$$\mathrm{Re}\left\{\frac{p(\alpha+\beta-\gamma)}{p-\eta} + (1+t)\left(1+\frac{zG''(z)}{G'(z)}\right)\right\} > 0.$$

Therefore, by virtue of Lemma 5, $L(z,t)$ is a subordination chain. We observe from the definition of subordination chain that

$$L(\zeta,t) \notin L(\mathrm{U},0) = \phi(\mathrm{U}) \quad (\zeta \in \partial\mathrm{U}; 0 \le t < \infty).$$

Now, suppose that F is not subordinate to G, then by Lemma 1, there exists points $z_0 \in \mathrm{U}$ and $\zeta_0 \in \partial\mathrm{U}$, such that

$$F(z_0) = G(\zeta_0) \ \text{ and } \ z_0 F(z_0) = (1+t)\zeta_0 G'(\zeta_0)$$

$$(0 \le t < \infty).$$

Hence, we have

$$L(\zeta_0,t) = G(\zeta_0) + \frac{(p-\eta)(1+t)}{p(\alpha+\beta-\gamma)}\zeta_0 G'(\zeta_0)$$

$$= F(z_0) + \frac{p-\eta}{p(\alpha+\beta-\gamma)} z_0 F'(z_0)$$

$$= \frac{(p-\eta)}{p} z_0^p Q^p_{\alpha_{-1},\beta,\gamma} f(z_0) + \frac{\eta}{p} z_0^p Q^p_{\alpha,\beta,\gamma} f(z_0) \in \phi(\mathrm{U}),$$

by virtue of the subordination condition (3.3). This contradicts the above observation $L(\zeta_0,t) \notin \phi(\mathrm{U})$. Therefore, the subordination condition (3.3) must imply the subordination given by (3.13). Considering $F(z) = G(z)$, we see that the function G is the best dominant. This evidently completes the proof of Theorem 1.

We next provide a dual problem of Theorem 1, in the sense that the subordinations are replaced by superordinations.

Theorem 2. Let

$$f, g \in \Sigma_{p,m},\ \alpha > \gamma,\ \beta > 0,\ \gamma > 0,\ p \in \mathrm{N},\ 0 \le \eta < p$$

and $z \in \mathrm{U}$. Suppose that

$$\mathrm{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta,$$

setting

$$\phi(z) := \frac{p-\eta}{p} z^p Q^p_{\alpha_{-1},\beta,\gamma} g(z) + \frac{\eta}{p} z^p Q^p_{\alpha,\beta,\gamma} g(z),$$

where δ is given by (3.2), and

$$\frac{p-\eta}{p} z^p Q^p_{\alpha_{-1},\beta,\gamma} f(z) + \frac{\eta}{p} z^p Q^p_{\alpha,\beta,\gamma} f(z),$$

is univalent in U and $z^p Q^p_{\alpha,\beta,\gamma} f(z) \in \mathrm{H}[1,1] \cap Q$. Then, the following superordination relation

$$\phi(z) \prec \frac{p-\eta}{p} z^p Q^p_{\alpha_{-1},\beta,\gamma} f(z) + \frac{\eta}{p} z^p Q^p_{\alpha,\beta,\gamma} f(z) \ (z \in \mathrm{U}), \tag{3.14}$$

implies that

$$z^p Q^p_{\alpha,\beta,\gamma} g(z) \prec z^p Q^p_{\alpha,\beta,\gamma} f(z) \qquad (z \in \mathrm{U}).$$

Moreover, the function $z^p Q^p_{\alpha,\beta,\gamma} g(z)$ is the best subordinant.

Proof. The first part of the proof is similar to that of Theorem 1 and so we will use the same notation as in the proof of Therem1.

Now, let us define the functions F and G by (3.5). We first note that, if the function q is defined by (3.6), using (3.7), then we obtain

$$\phi(z) = G(z) + \frac{p-\eta}{p(\alpha+\beta-\gamma)} zG' := \varphi(G(z), zG'(z)).$$

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(3.15)

Then using the same method as in the proof of Theorem 1. We can prove that

$$\mathrm{Re}\{q(z)\} > 0 \quad (z \in \mathrm{U}),$$

that is, G defined by (3.5) is convex (univalent) in U .
Next, we prove that the subordination condition (3.14) implies that

$$G(z) \prec F(z) \quad (z \in \mathrm{U}) \tag{3.16}$$

for the functions F and G defined by (3.5). Now, consider the function $L(z,t)$ defined by

$$L(z,t) := G(z) + \frac{(p-\eta)t}{p(\alpha+\beta-\gamma)} zG'(z)(0 \le t < \infty;\ z \in \mathrm{U}).$$

As G is convex and $p(\alpha+\beta-\gamma)/(p-\eta) > 0,$ we can prove easily that $L(z,t)$ is a subordination chain as in the proof of Theorem 1. Therefore, according to Lemma 2, we conclude that the superordination condition (3.14) must imply the superordination given by (3.16). Furthermore, as the differential equation (3.15) has the univalent solution G , it is the best subordinate of the given differential superordination. Therefore, we complete the proof of Theorem 2. If we combine Theorems 1 and 2, then we obtain the following sandwich -type theorem.

Theorem 3. Let $f, g_k \in \Sigma_{p,m}$ $(k=1,2),\ \alpha > \gamma,\ \beta > 0,\ \gamma > 0,$ $p \in \mathrm{N},\ 0 \le \eta < p$ and $z \in \mathrm{U}.$ Suppose that

$$\mathrm{Re}\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\delta, \tag{3.17}$$

setting

where δ is given by (3.2), and

$$\frac{p-\eta}{p} z^p Q^p_{\alpha_{-1},\beta,\gamma} f(z) + \frac{\eta}{p} z^p Q^p_{\alpha,\beta,\gamma} f(z),$$

is univalent in U and $z^p Q^p_{\alpha,\beta,\gamma} f(z) \in \mathbf{H}[1,1] \cap Q.$ Then, the following relation

$$\phi_1(z) \prec \frac{p-\eta}{p} z^p Q^p_{\alpha_{-1},\beta,\gamma} f(z) + \frac{\eta}{p} z^p Q^p_{\alpha,\beta,\gamma} f(z) \prec \phi_2(z) \quad (z \in \mathrm{U})$$

Implies that

Moreover, the function $z^p Q^p_{\alpha,\beta,\gamma} g_1(z)$ and $z^p Q^p_{\alpha,\beta,\gamma} g_2(z)$ are the best subordinate and the best dominant, respectively.

The assumption of Theorem 3, that the functions

need to be univalent in U , may be replaced by another condition in the following result.

Corollary 1. Let $f, g_k \in \Sigma_{p,m}$ $(k=1,2),\ \alpha > \gamma,\ \beta > 0,\ \gamma > 0,\ p \in \mathrm{N},$ $0 \le \eta < p$ and $z \in \mathrm{U}.$ Suppose that condition (3.17) is satisfied and

$$\mathrm{Re}\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > -\delta, \tag{3.18}$$

setting

$$\psi(z) := \frac{p-\eta}{p} z^p Q^p_{\alpha_{-1},\beta,\gamma} f(z) + \frac{\eta}{p} z^p Q^p_{\alpha,\beta,\gamma} f(z),$$

where δ is given by (3.2). Then, the following relation

$$\phi_1(z) \prec \frac{p-\eta}{p} z^p Q^p_{\alpha_{-1},\beta,\gamma} f(z) + \frac{\eta}{p} z^p Q^p_{\alpha,\beta,\gamma} f(z) \prec \phi_2(z) \quad (z \in \mathrm{U}),$$

implies that

Moreover, the function $z^p Q^p_{\alpha,\beta,\gamma} g_1(z)$ and $z^p Q^p_{\alpha,\beta,\gamma} g_2(z)$ are the best subordinate and the best dominant, respectively.

Proof. To prove Corollary 1, we have to show that condition (3.18) implies univalent of $\psi(z)$ and

$$F(z) := z^p Q^p_{\alpha,\beta,\gamma} g_1(z).$$

As $0 < \delta \le 1/2$ from Theorem 1, condition (3.18) means that ψ is a close- to-convex function in U (see [4]) and hence ψ is univalent in U . Furthermore, using the same techniques as in the proof of Theorem 1, we can prove the convexity (univalent) of F and so the details may be omitted here. Therefore, by applying Theorem 3, we obtain Corollary1.

Theorem 4. Let $f, g_k \in \Sigma_{p,m}$ $(k=1,2),\ \alpha > (p-\eta)/p,\ 0 \le \eta < p$ and $z \in \mathrm{U}.$ Suppose that

$$\mathrm{Re}\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\delta, \tag{3.19}$$

setting

$$\phi_k(z) := \frac{p-\eta}{p} z^{p+1} Q^p_{\alpha_{-1},\beta,\gamma}\ g_k(z) + \frac{\eta}{p} z^{p+1} Q^p_{\alpha,\beta,\gamma}\ g_k(z),$$

where

$$\delta = \frac{(p-\eta)^2 + (p(\alpha+\beta-\gamma-1)+\eta)^2}{4(p-\eta)(p(\alpha+\beta-\gamma-1)+\eta)} - \frac{\left|(p-\eta)^2 - (p(\alpha+\beta-\gamma-1)+\eta)^2\right|}{4(p-\eta)(p(\alpha+\beta-\gamma-1)+\eta)},$$

and

$$\frac{p-\eta}{p} z^{p+1} Q^p_{\alpha_{-1},\beta,\gamma}\ f(z) + \frac{\eta}{p} z^{p+1} Q^p_{\alpha,\beta,\gamma}\ f(z),$$

Is univalent in U and $z^{p+1} Q^p_{\alpha,\beta,\gamma} f(z) \in \mathbf{H}[0,1] \cap Q$.
Then, the following relation

$$\phi_1(z) \prec \frac{p-\eta}{p} z^{p+1} Q^p_{\alpha_{-1},\beta,\gamma}\ f(z) + \frac{\eta}{p} z^{p+1} Q^p_{\alpha,\beta,\gamma}\ f(z) \prec \phi_2(z) \quad (z \in \mathrm{U}),$$

Implies that

$$z^{p+1} Q^p_{\alpha,\beta,\gamma} g_1(z) \prec z^{p+1} Q^p_{\alpha,\beta,\gamma} f(z) \prec z^{p+1} Q^p_{\alpha,\beta,\gamma} g_2(z) \quad (z \in \mathrm{U}).$$

Moreover, the functions $z^{p+1} Q^p_{\alpha,\beta,\gamma} g_1(z)$ and dominant, respectively.

Next, we consider the integral operator $F_\mu\ (\mu > 0)$ defined by (ef. [5],[12])

$$F_\mu(f)(z) := \frac{\mu}{z^{\mu+p}} \int_0^z t^{\mu+p-1} f(z)dt \quad (f \in \Sigma_{p,m};\ \mu > 0) \tag{3.20}$$

Now, we obtain the following result involving the integral operator defined by (3.20).

Theorem 5. Let $f, g_k \in \Sigma_{p,m}$ $(k = 1,2)$. Suppose also that

$$\mathrm{Re}\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\delta \tag{3.21}$$

Setting

$$\phi_k(z) := z^p Q^p_{\alpha,\beta,\gamma} g_k(z); k = 1,2; z \in \mathrm{U},$$

where

$$\delta = \frac{1+\mu^2 - \left|1-\mu^2\right|}{4\mu} \quad (\mu > 0), \tag{3.22}$$

and $z^p Q^p_{\alpha,\beta,\gamma} f(z)$ is univalent in U and $z^p Q^p_{\alpha,\beta,\gamma}\ F_\mu(z) \in \mathbf{H}[1,1] \cap Q$. Then, the following relation

$$\phi_1(z) \prec z^p Q^p_{\alpha,\beta,\gamma} f(z) \prec \phi_2(z) \quad (z \in \mathrm{U})$$

Implies that

$$z^p Q^p_{\alpha,\beta,\gamma} F_\mu(g_1)(z) \prec z^p Q^p_{\alpha,\beta,\gamma} F_\mu(f)(z) \prec z^p Q^p_{\alpha,\beta,\gamma} F_\mu(g_2)(z) \quad (z \in \mathrm{U}).$$

Moreover, the functions $z^p Q^p_{\alpha,\beta,\gamma} F_\mu(g_1)(z)$ and $z^p Q^p_{\alpha,\beta,\gamma} F_\mu(g_2)(z)$ are the best subordinate and the best dominant, respectively.

Proof. Let us define the functions F and G_k $(k = 1,2)$ by

$$F(z) := z^p Q^p_{\alpha,\beta,\gamma} F_\mu(f)(z)$$

and $G_k(z) := z^p Q^p_{\alpha,\beta,\gamma} F_\mu(g_k)(z),$

respectively. Without loss of generality, as in the proof of Theorem 1, we can assume that G_k is analytic and univalent on $\overline{\mathrm{U}}$, and

$$G_k'(\zeta) \neq 0 \quad (\zeta \in \partial\mathrm{U}).$$

From the definition of the integral operator F_μ defined by (3.20), we obtain

$$z(Q^p_{\alpha,\beta,\gamma} F_\mu(f)(z))' = \mu Q^p_{\alpha_{-1},\beta,\gamma} f(z) - (\mu + p) Q^p_{\alpha,\beta,\gamma} F_\mu(f)(z). \tag{3.23}$$

Then, from (3.21) and (3.23), we have

$$\mu\phi_k(z) = \mu G_k(z) + zG_k'(z). \tag{3.24}$$

Setting

$$q_k(z) = 1 + \frac{zG_k''(z)}{G_k'(z)} \quad (k = 1,2;\ z \in \mathrm{U}), \tag{3.25}$$

And differentating both sides of (3.24), we obtain

$$1 + \frac{z\phi_k''(z)}{\phi_k'(z)} = q_k(z) + \frac{zq_k'(z)}{q_k(z) + \mu}.$$

The remaining part of the proof is similar to that of Theorem 1 and so is omitted the proof involved.

REFERENCES

[1] M. K. Aouf, A. Shamandy, A. O. Mostafa and F. Z. El-Emam, "On certain subclasses of meromorphically p-valent functions associated with integral operators", European J. Pure Appl. Math., 4 (2011), no. 4, 435-447.

[2] E. Aqlan, J. M. Jahangiri and S. R. Kulkarni, "Certain integral operators applied to meromorphic p-valent functions", J. Nat. Geom., 24 (2003), 111-120.

[3] R. M. El-Ashwah, M. K. Aouf and Ahmed M. Abd-Eltawab, "Applications of differential subordination on certain subclasses of p-valent meromorphic functions", An. Univ. Oradea Fasc. Mat., 20 (2013), no. 1, 125-136.

[4] W. Kaplan, "Close-to-convex schlicht functions", Michigan Math. J., 2 (1952), 169-185.

ISSN 2229-5518

[5] V. Kumar and S. L. Shukla, "Certain integrals for classes of p-valent functions", Bull. Austral Math. Soc., 25 (1982), 85-97.

[6] A. Y. Lashin, "On certain subclasses of meromorphic functions associated with certain integral operators", Comput. Math. Appl., 5 (2010), 524-531.

[7] S. S. Miller and P. T. Mocanu, "Differential Subordination and univalent functions", Michigan Math. J., 28 (1981), 157 -171.

[8] S. S. Miller and P. T. Mocanu, "Univalent solutions of Briot-Bouquet differential equations", J. Different Eq., 57 (1985), 297-309.

[9] S. S. Miller and P. T. Mocanu, "Differential Subordination. Theory and Applications", Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker, New York and Basel, 2000.

[10] S. S. Miller and P. T. Mocanu, "Subordinates of differential superordinations", Complex Var Theory Appl, 48 (2003), 815-826.

[11] C. Pommerenke, "Univalent functions", Vanderhoeck and Ruprech, Göttingen, 1975.

[12] H. M. Srivastava and S. Owa, "Current Topics in Analytic Function Theory", World Scientific Publishing Company, Singapore, New Jersey, London, Hong Kong, 1992.